Solution of Ordinary Differential Equations

Dr. Fatimah Sahib Kadhim Al-Taie

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Chapter 1

Introduction

1.1 Basic Concepts and Definitions

Differential Equations: The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities. Because the derivative $f^{'}(t)$ of the function f is the rate at which the quantity $f(t)$ is changing with respect to the independent variable[1] t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a differential equation, (DE).

A DE is used to describe changing quantities and it plays a major role in qualitative studies in many disciplines such as all areas of engineering, physical sciences, life sciences, and economics.

The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x^{'}(t) = \frac{dx}{dt}$.
The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y.

Examples:
Are they DEs or not?

$ax^2 + bx + c = 0$	No!
$ax^2 + bx^{'} + c = 0$	Yes! Here $x^{'} = \frac{dx}{dt}$

[1] When an equation involves one or more derivatives with respect to a particular variable,that variable is called an independent variable. A variable is called dependent if a derivative of that variable is occurs.

$ax^2 + bx' + cy' = 0$ Yes! Here $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dx}$
$y'' = x^3$ Yes! Here $y'' = \frac{d^2y}{dx^2}$.

Applications of Differential Equations:
As mentioned before, in real life there are a lot of applications of DEs such as e.g. Newton's law of cooling, where the physical law can be translated into a differential equation. Another examples are Torricelli's law, heat transfer in materials, relaxation in nuclear magnetic resonance, radioactive decay, and chemical reaction kinetics where the rate law or rate equation for a chemical reaction is a DE that links the reaction rate with concentrations or pressures of reactants and constant parameters.

1.2 Classification of Differential Equation

There are some ways of classifying differential equations.

1. Order and Degree of DEs: The <u>order</u> of a DE is determined by the highest order derivative of the dependent variable.

   Examples: Determine the order of the following DEs:

$$ax + \frac{dx}{dt} = 0 \qquad \text{"first order DE"}$$

$$\frac{dx}{dt} + ax^2 = 0 \qquad \text{"first order DE"}$$

$$\frac{d^2x}{dt^2} + bx = 0 \qquad \text{"second order DE"}$$

$$bx^4 + \frac{d^2x}{dt^2} = 0 \qquad \text{"second order DE"}$$

$$\frac{d^{(n)}x}{dt^{(n)}} + cx = 0 \qquad \text{"n − th order DE".}$$

   While the <u>degree</u> of a DE is determined by the power of the highest order derivative present in the equation.
   Examples:

$$(\frac{d^2x}{dt^2})^5 + b(\frac{d^3x}{dt^3})^2 = 0 \qquad \text{"second degree third order DE"}$$

$$\frac{dx}{dt} + ax^3 = 0 \qquad \text{"first degree first order DE"}$$

$$(\frac{d^2x}{dt^2})^{(n)} + a\frac{dx}{dt} + bx = 0 \qquad \text{"n − th degree second order DE".}$$

2. Homogeneous and Nonhomogeneous DEs: A differential equation is nonhomogeneous if it has terms involving only the independent variable (and constants) on the right hand side, and it is homogeneous if this right hand side is zero.
Examples: The differential equations

$$\frac{dy}{dx} + x^2 y = 4x^3$$

$$\frac{d^4y}{dx^4} + x\frac{d^2y}{dx^2} + y^2 = 6x + 3$$

are nonhomogeneous.
While the following differential equation

$$\frac{d^4y}{dx^4} + x\frac{d^2y}{dx^2} + y^2 = 0.$$

is homogeneous.

3. Linear and Nonlinear DEs: A crucial classification of differential equations is whether they are linear or nonlinear. The differential equation is said to be linear if it is a linear function in the dependent variable y and its derivatives, but not in x.

Examples: The differential equation

$$e^x \frac{d^2y}{dx^2} + (cosx)\frac{dy}{dx} + (1 - \sqrt{x})y = tan^{-1}x$$

is linear because the dependent variable y and its derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ appear linearly. By contrast, the equations

$$\frac{d^2y}{dx^2} = y\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + 3(\frac{dy}{dx})^2 + 4y^3 = 0$$

are not linear because products and powers of y or its derivatives appear. Also,

$$\frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^3 + y = 0 \qquad \text{is nonlinear}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 4x^3 \qquad \text{is linear}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \qquad \text{is linear.}$$

4. Ordinary and Partial Differential Equations: One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an *Ordinary Differential Equation*, (ODE). In the second case, the derivatives are partial derivatives, and the equation is called a *Partial Differential Equation* (PDE).

   Examples: All the above differential equations, which were discussed before, are ODEs. Another example of ODE is

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + x(t) = 0 \qquad \text{"ODE"}$$

   While examples of PDEs are

$$\frac{\partial^2 y(t,s)}{\partial s^2} - \frac{\partial y(t,s)}{\partial t} = 0 \qquad \text{"PDE"}$$

$$\frac{\partial^2 y(t,s)}{\partial s^2} - \frac{\partial^2 y(t,s)}{\partial t^2} = 0 \qquad \text{"PDE"}$$

   In this course, only ODEs are considered.

1.3 Solutions of ODE

In algebra, we typically seek the unknown numbers that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we are challenged to find the unknown function $y = y(x)$ for which an identity such as

$$\frac{dy}{dx} = 2xy$$

holds on some interval of real numbers. Ordinarily, we will want to find all solutions of the differential equation, if possible.
Example:

$$ay^{'}(x) + b = 0$$

$$y^{'} = -\frac{b}{a} \qquad \text{is not a solution}$$

$$y = -\int \frac{b}{a}\, dx \qquad \text{is a solution.}$$

Example: Show that $y = e^{2x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$$

Solution:

$$y = e^{2x} \quad \Longrightarrow \frac{dy}{dx} = 2\ e^{2x} \Longrightarrow \frac{d^2y}{dx2} = 4\ e^{2x};$$

then

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 4\ e^{2x} + 2\ e^{2x} - 6\ e^{2x} = 0.$$

There are some methods for finding the solution of the differential equations, these methods are discussed in the following next chapters.

Chapter 2

Methods of Finding the Solution of the First Order Differential Equation

There are some methods of finding the solution of the first order ODE based on integration as general and particular solutions.

2.1 Separation of Variables

The first order differential equation

$$\frac{dy}{dx} = H(x, y). \tag{2.1}$$

is called separable provided that $H(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = M(x)\,\frac{1}{N(y)}, \tag{2.2}$$

In this case the variables x and y can be *separated* (isolated on opposite sides of an equation) by writing informally the equation

$$N(y)\,dy = M(x)\,dx. \tag{2.3}$$

It is easy to solve this special type of differential equation simply by integrating the right hand side with respect to x and the left hand side with respect to y as follows:

$$\int_y N(y)\,dy = \int_x M(x)\,dx. \tag{2.4}$$

which gives

$$H_1(y) = H_2(x) + c. \tag{2.5}$$

where $H_1(y)$ and $H_2(x)$ are the antiderivatives of $N(y)$ and $M(x)$, respectively.
Eq.(2.5) is a general solution of eq.(2.1), meaning that it involves an arbitrary constant

c, and for every choice of c it is a solution of the differential equation in (2.1).

Example 1. Find the solution of

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}. \tag{2.6}$$

Solution: The given eq. in (2.6) is separable. Rewrite it as

$$(1-y^2)\, dy = x^2\, dx.$$

In order to solve the above equation, we have to integrate the left hand side with respect to y while the right hand side with respect to x

$$\int (1-y^2)\, dy = \int x^2\, dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$
$$3y - y^3 = x^3 + 3c.$$

where c is an arbitrary constant.

Note: In eq.(2.1), if in addition to the differential equation, an initial condition

$$y(x_0) = y_0$$

is prescribed, then the solution of eq.(2.1) satisfying this condition is obtained by setting $x = x_0$ and $y = y_0$ in eq.(2.5). This gives

$$H_1(y_0) = H_2(x_0) + c$$

then $c = H_1(y_0) - H_2(x_0)$. Substitute in eq.(2.5) gives the particular solution of eq.(2.1).

Example 2. Solve the following initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1.$$

Solution: The differential equation can be written as

$$2(y-1)dy = (3x^2+4x+2)dx.$$

Integrate the left hand side with respect to y and the right hand side with respect to x gives

$$\int 2(y-1)dy = \int (3x^2+4x+2)dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c \tag{2.7}$$

where c is an arbitrary constant.
In order to determine the value of c in eq.(2.7), we substitute the given initial condition $(y(0) = -1)$ in eq.(2.7), obtaining $c = 3$.
Hence, the particular solution of the initial value problem is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

Example 3. Solve the initial value problem

$$\frac{dy}{dx} = -6xy, \qquad y(0) = 7.$$

Solution: Divide both sides of the DE by y and multiply each side by dx to get

$$\frac{dy}{y} = -6x\ dx. \tag{2.8}$$

Then integrate both sides

$$\int \frac{dy}{y} = \int (-6x)\ dx$$

$$ln\ y = -3x^2 + c$$

$$y = e^{-3x^2+c} = e^{-3x^2} e^c \tag{2.9}$$

let

$$e^c = A$$

then eq.(2.9) is redefined as

$$y = Ae^{-3x^2} \tag{2.10}$$

substitute the initial condition $(y(0) = 7)$ in eq.(2.10) yields $A = 7$.
So, the particular solution is

$$y = 7e^{-3x^2}.$$

Example 4.
Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y\ cosx}{1 + 2y^2}, \qquad y(0) = 1$$

Solution: Rewrite the DE as $\frac{1 + 2y^2}{y}\ dy = cosx\ dx$, such that the resulting equation is separable. Hence

$$\int \frac{1 + 2y^2}{y}\ dy = \int cosx\ dx$$

$$ln\ y + y^2 = sinx + c.$$

Substitute the given initial condition, gives $c = 1$. Hence, the solution of the DE is

$$ln\ y + y^2 = sinx + 1.$$

Application in Reality

Cooling and Heating Equation:
Newton's law of cooling may be stated in this way:
The time rate of change of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium. That is

$$\frac{dT}{dt} = k(A - T) \tag{2.11}$$

where k is a positive constant. Observe that if $T > A$, then $\frac{dT}{dt} < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $\frac{dT}{dt} > 0$, so that T is increasing.
Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then –with the aid of this formula– we can predict the future temperature of the body.

Example 5.
A roast, initially at 50F, is placed in a 375F oven at 5:00 Pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125F. When will the roast be 150F (medium rare)?

Solution: We take t in minutes, with $t = 0$ corresponding to 5:00 pm.
We have $T(t) < A = 375$, $T(0) = 50$, and $T(75) = 125$. Hence

$$\frac{dT}{dt} = k(375 - T)$$
$$\int \frac{dT}{375 - T} = \int k\ dt$$
$$-ln(375 - T) = kt + c$$
$$375 - T = Be^{-kt}.$$

Now, $T(0) = 50$ implies that $B = 325$, so $T(t) = 375 - 325e^{-kt}$. We also know that $T(75) = 125$. Substitution of these values in the preceding equation yields

$$k = -\frac{1}{175} ln(\frac{250}{325}) \approx 0.0035.$$

Hence we finally solve the equation

$$150 = 375 - 325e^{(-0.0035)t}$$

for $t = -\dfrac{ln(225/325)}{0.0035} \approx 105$ min, the total cooking time required. Because the roast was placed in the oven at 5:00 pm, it should be removed at about 6:45 pm.

Exercises:

A. Find the particular solution of

1. $2xy\,\dfrac{dy}{dx} = 1 + y^2, \qquad y(2) = 3.$
2. $xy^2 dx + e^x dy = 0, \qquad y(0) = \dfrac{1}{2}.$

B. Find the general solution of the following

1. $(1-x)\dfrac{dy}{dx} = y^2.$
2. $sinx\ cosy\ dx + cosx\ siny\ dy = 0.$
3. $ye^{2x}\ dx = (4 + e^{2x})\ dy.$

2.2 Equations with Homogeneous Coefficients

Homogeneous functions: The function $f(x,y)$ is said to be homogeneous of degree k in x and y if, and only if,

$$f(\lambda x, \lambda y) = \lambda^k f(x,y).$$

The definition is easily extended to functions of more than two variables.

Examples:

1. $f(x,y) = 2y^3\ exp(\frac{y}{x}) - \frac{x^4}{x+3y}$

$f(\lambda x, \lambda y) = 2\lambda^3 y^3\ exp(\frac{\lambda y}{\lambda x}) - \frac{\lambda^4 x^4}{\lambda x + 3\lambda y} = \lambda^3 f(x,y);$
hence $f(x,y)$ is homogeneous of degree 3 in x and y.

2. $f(x,y) = \sqrt{x+4y}$

$f(\lambda x, \lambda y) = \sqrt{\lambda x + 4\lambda y} = \lambda^{\frac{1}{2}}\ f(x,y);$
hence the function $f(x,y) = \sqrt{x+4y}$ is homogeneous of degree $\frac{1}{2}$ in x and y.

3. $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$

$f(\lambda x, \lambda y) = \frac{\lambda x}{\sqrt{\lambda^2 x^2 + \lambda^2 y^2}} = \lambda^0 f(x,y);$
hence the function $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ is homogeneous of degree zero in x and y.

To solve equations with homogeneous coefficients, suppose the coefficients M and N in an equation of order one,

$$M(x,y)dx + N(x,y)dy = 0 \tag{2.12}$$

are both homogeneous functions and are of the same degree in x and y. The ratio $\frac{M}{N}$ is a function of $\frac{y}{x}$ alone.
Hence eq.(2.12) may be put in the form

$$\frac{dy}{dx} + g(\frac{y}{x}) = 0 \tag{2.13}$$

This suggest the introduction of a new variable v by putting $y = vx$ and $\frac{dy}{dx} = v\ dx + x\ dv$.
Then (2.13) becomes

$$x\frac{dv}{dx} + v + g(v) = 0, \tag{2.14}$$

in which the variables are *separable*. We can obtain the solution of (2.14) by the separation of variable method in the previous section. After finding the solution, we have to back to

the original variables by inserting $\frac{y}{x}$ for v, and thus arrive at the solution of (2.12).
We have shown that the substitution $y = vx$ will transform eq.(2.12) into an equation in v and x in which the variables are separable.
Note: The above method would also be successful if we used $x = vy$ (instead of $y = vx$) to obtain from (2.12) an equation in y and v (instead of an equation of x and v).

Example 1. Solve the equation

$$(x^2 - xy + y^2)\ dx - xy\ dy = 0. \tag{2.15}$$

Solution: Since the coefficients in (2.15) are both homogeneous and of degree two in x and y, let us put $y = vx$, $\quad dy = v\ dx + x\ dv$.
Then (2.15) becomes

$$(\quad (x^2 - x^2v + x^2v^2)\ dx - x^2v(v\ dx + x\ dv) = 0, \quad) * \frac{1}{x^2}$$

$$(1 - v + v^2)\ dx - v(v\ dx + x\ dv) = 0$$

$$(1 - v)\ dx + v^2\ dx - v^2\ dx - vx\ dv = 0.$$

$$(\quad (1 - v)\ dx - xv\ dv = 0. \quad) * \frac{-1}{x(v-1)}$$

Hence we separate variables to get

$$\frac{dx}{x} + \frac{v\ dv}{v-1} = 0$$

$$\int_x \frac{dx}{x} + \int_v (1 + \frac{1}{v-1})dv = 0$$

$$ln\ x + v + ln(v-1) = ln\ c$$

or

$$x(v-1)\ e^v = c.$$

In terms of the original variables, the solution is given by

$$x(\frac{y}{x} - 1)\ exp(\frac{y}{x}) = c$$

or

$$(y - x)\ exp(\frac{y}{x}) = c.$$

Example 2. Solve the equation

$$xy\ dx + (x^2 + y^2)\ dy = 0 \tag{2.16}$$

Solution: Again the coefficients in the equation are homogeneous and of degree two.
We could use $y = vx$, but the relation simplicity of the dx term in (2.16) suggests that we put $x = vy$. Then $dx = v\ dy + y\ dv$, and equation (2.16) is replaced by

$$(\quad vy^2(v\ dy + y\ dv) + (v^2y^2 + y^2)\ dy = 0, \quad) * \frac{1}{y^2}$$

$$v(v\ dy + y\ dv) + (v^2 + 1)\ dy = 0.$$

Hence we need to solve

$$(\quad vy\ dv + (2v^2 + 1)\ dy = 0, \quad) * \frac{1}{y(2v^2+1)}$$

we separate the variables such that

$$\int \frac{v}{2v^2+1}\ dv + \int \frac{dy}{y} = 0$$

$$ln(2v^2 + 1) + 4\ ln\ y = ln\ c$$

$$y^4(2v^2 + 1) = c.$$

Thus the desired solution is given by

$$y^4(\frac{2x^2}{y^2} + 1) = c;$$

that is

$$y^2(2x^2 + y^2) = c.$$

Remark: It is quite immaterial whether one uses $y = vx$ or $x = vy$. However, it is sometimes easier to substitute for the variable whose differential has the simpler coefficient.

Example 3. Check whether the following differential equation has homogeneous coefficients of the same degree or not; then find its solution.

$$y\ dx = (x + \sqrt{y^2 - x^2})\ dy. \tag{2.17}$$

Solution: The coefficients in the equation are homogeneous and of degree one.
Let

$$x = v\ y, \qquad dx = v\ dy + y\ dv. \tag{2.18}$$

Put (2.18) in (2.17), yields

$$y\ (v\ dy + y\ dv) = (vy + \sqrt{y^2 - v^2\ y^2})\ dy$$

$$yv\ dy + y^2 dv = (vy + y\ \sqrt{1 - v^2})\ dy$$

$$yv\ dy + y^2 dv = vy\ dy + (y\ \sqrt{1 - v^2})\ dy$$

$$y^2 dv = y\ \sqrt{1 - v^2}\ dy$$

$$(\quad y^2 dv = y\ \sqrt{1 - v^2}\ dy \quad) * \frac{1}{y^2\ \sqrt{1 - v^2}}$$

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dy}{y}$$

$$sin^{-1}(v) = ln\ y + c.$$

But $v = \frac{x}{y}$, so the solution is

$$sin^{-1}(\frac{x}{y}) = ln\ y + c.$$

Exercises:

Find the solution of the following
1. $(x - 2y)\ dx + (2x + y)\ dy = 0.$

2. $y\ dx = (x + \sqrt{y^2 - x^2})\ dy$

3. $(y - \sqrt{x^2 + y^2})\ dx - x\ dy = 0; \quad y(\sqrt{3}) = 1.$

4. $(3x^2 - 2y^2)\ y^{'} = 2xy; \quad y(0) = -1.$

2.3 Linear First-Order Equation

Previously, we saw how to solve a separable differential equation by integrating after multiplying both sides by an appropriate factor.
For instance, to solve the equation

$$\frac{dy}{dx} = 2xy \tag{2.19}$$

we multiply both sides by the factor $\dfrac{1}{y}$ to get

$$\frac{1}{y}\,\frac{dy}{dx} = 2x;$$

that is

$$\frac{dy}{y} = 2x\ dx. \tag{2.20}$$

Because each side of the equation in (2.20) is recognizable as a derivative, all that remains are two simple integrations, which yield $ln\ y = x^2 + c$. For this reason, the function $\mu(y) = \dfrac{1}{y}$ is called an integrating factor for the original equation in (2.19).
An integrating factor for a differential equation is a function $\mu(x, y)$ such that the multiplication of each side of the differential equation by $\mu(x, y)$ yields an equation in which each side is recognizable as a derivative.
With the aid of the appropriate integrating factor, there is a standard technique for solving the linear first-order equation

$$\frac{dy}{dx} + p(x)y = Q(x), \tag{2.21}$$

On an interval on which the coefficient functions $p(x)$ and $Q(x)$ are continuous. We multiply each side in eq.(2.21) by the integrating factor

$$\mu(x) = e^{\int p(x)\ dx}. \tag{2.22}$$

The result is

$$e^{\int p(x)\ dx}\,\frac{dy}{dx} + p(x)\ e^{\int p(x)\ dx}\ y = Q(x)\ e^{\int p(x)\ dx}. \tag{2.23}$$

Because

$$\frac{d}{dx}\Big(\int p(x)\ dx\Big) = p(x),$$

the left-hand side is the derivative of the product $y(x)\ e^{\int p(x)\ dx}$, so eq.(2.23) is equivalent to

$$\frac{d}{dx}\Big(y(x)\ e^{\int p(x)\ dx}\Big) = Q(x)\ e^{\int p(x)\ dx}\ ;$$

$$d\Big(y(x)\ e^{\int p(x)\ dx}\Big) = \Big(Q(x)\ e^{\int p(x)\ dx}\Big)\ dx.$$

Integration of both sides of this equation gives

$$y(x)\ e^{\int p(x)\ dx} = \int\Big(Q(x)e^{\int p(x))\ dx}\Big)dx + c.$$

Finally, solving for y,we obtain the general solution of the linear first-order equation in (2.21):

$$y(x) = e^{-\int p(x)\,dx}\left(\int (Q(x)e^{\int p(x))\,dx})dx + c\right). \tag{2.24}$$

That is, in order to solve an equation that can be written in the form in eq.(2.21) with the coefficient functions $p(x)$ and $Q(x)$ displayed explicitly, you should attempt to carry out the following steps.
Method: Solution of First-order equations

1. Begin by calculating the integrating factor $\mu(x) = e^{\int p(x)\,dx}$.
2. Then multiply both sides of the differential equation by $\mu(x)$.
3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:
$$\frac{d}{dx}\Big(\mu(x)\,y(x)\Big) = \mu(x)\,Q(x).$$
4. Finally, integrate this equation,
$$\mu(x)\,y(x) = \int \mu(x)\,Q(x)\,dx + c,$$
then solve for y to obtain the general solution of the original differential equation.

Remark: Given an initial $y(x_0) = y_0$, you can (as usual) substitute $x = x_0$ and $y = y_0$ into the general solution and solve for the value of c yielding the particular solution that satisfies this initial condition, where c is a unique value.
Consequently, we have shown the following existence-uniqueness theorem.

Theorem 2.1 *If the functions $p(x)$ and $Q(x)$ are continuous on an open interval containing the point x_0, then the initial value problem*

$$\frac{dy}{dx} + p(x)\,y = Q(x), \qquad y(x_0) = y_0 \tag{2.25}$$

has a unique solution $y(x)$ on the interval, given by the formula in eq.(2.24) with an appropriate value of c.

Example 1. Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8}\,e^{\frac{-x}{3}}, y(0) = -1. \tag{2.26}$$

solution: Here we have $p(x) = -1$ and $Q(x) = \dfrac{11}{8}\,e^{\frac{-x}{3}}$, so the integrating factor is $\mu(x) = e^{\int(-1)\,dx} = e^{-x}$.
Multiplication of both sides of the given equation (2.26) by e^{-x} yields

$$e^{-x}\,\frac{dy}{dx} - e^{-x}\,y = \frac{11}{8}\,e^{\frac{-4x}{3}}, \tag{2.27}$$

which we recognize as

$$\frac{d}{dx}\Big(e^{-x}\, y\Big) = \frac{11}{8}\, e^{\frac{-4x}{3}}.$$

Hence integration with respect to x gives

$$e^{-x}\, y = \int \frac{11}{8}\, e^{\frac{-4x}{3}}\, dx = \frac{-33}{32}\, e^{\frac{-4x}{3}} + c,$$

and multiplication by e^{x} gives the general solution

$$y(x) = ce^{x} - \frac{33}{32}\, e^{\frac{-x}{3}}.$$

Substitution of $x = 0$ and $y = -1$ now gives $c = \dfrac{1}{32}$, so the desired particular solution is

$$y(x) = \frac{1}{32}\, e^{x} - \frac{33}{32}\, e^{\frac{-x}{3}} = \frac{1}{32}\,\Big(e^{x} - 33\, e^{\frac{-x}{3}}\Big).$$

Example 2. Find a general solution of

$$(x^2+1)\,\frac{dy}{dx} + 3xy = 6x. \tag{2.28}$$

Solution: After division of both sides of the equation by x^2+1, we recognize the result

$$\frac{dy}{dx} + \frac{3x}{x^2+1}\, y = \frac{6x}{x^2+1}$$

as a first-order linear equation with $p(x) = \dfrac{3x}{x^2+1}$ and $Q(x) = \dfrac{6x}{x^2+1}$. Multiplication by

$$\mu(x) = exp\bigg(\int \frac{3x}{x^2+1}\, dx\bigg) = exp\bigg(\frac{3}{2}\, ln(x^2+1)\bigg) = (x^2+1)^{\frac{3}{2}}$$

yields

$$(x^2+1)^{\frac{3}{2}}\, \frac{dy}{dx} + 3x\,(x^2+1)^{\frac{1}{2}}\, y = 6x\,(x^2+1)^{\frac{1}{2}},$$

and thus

$$\frac{d}{dx}\Big((x^2+1)^{\frac{3}{2}}\, y\Big) = 6x\,(x^2+1)^{\frac{1}{2}}.$$

Integration then yields

$$(x^2+1)^{\frac{3}{2}}\, y = \int 6x\,(x^2+1)^{\frac{1}{2}}\, dx = 2(x^2+1)^{\frac{3}{2}} + c.$$

Multiplication of both sides by $(x^2+1)^{\frac{-3}{2}}$ gives the general solution

$$y(x) = 2 + c\,(x^2+1)^{\frac{-3}{2}}.$$

Example 3. Find the general solution of

$$y' = 1 + x + y + x\,y.$$

Solution:

$$\frac{dy}{dx} = 1 + x + y + x\,y$$

$$\frac{dy}{dx} - (1+x)\,y = 1 + x, \tag{2.29}$$

which is a linear first-order equation with

$$p(x) = -(1+x), \quad Q(x) = 1 + x.$$

Multiply (2.29) by $\mu(x) = e^{\int -(1+x)\,dx} = e^{-(x+\frac{x^2}{2})}$, yields

$$e^{-(x+\frac{x^2}{2})}\frac{dy}{dx} - e^{-(x+\frac{x^2}{2})}\,(1+x)\,y = e^{-(x+\frac{x^2}{2})}\,(1+x)$$

and thus

$$\frac{d}{dx}\,(e^{-(x+\frac{x^2}{2})}\,y) = e^{-(x+\frac{x^2}{2})}\,(1+x).$$

Integration then gives

$$y\,e^{-(x+\frac{x^2}{2})} = \int ((1+x)\,e^{-(x+\frac{x^2}{2})})\,dx.$$

$$y = -1 + e^{(x+\frac{x^2}{2})}\,c.$$

Mixture Problems: As an application of linear first-order equations, we consider a tank containing a solution (a mixture of solute and solvent) such as salt dissolved in water. There is both inflow and outflow, and we want to compute the amount $x(t)$ of solute in the tank at time t, given the amount $x(0) = x_0$ at time $t = 0$. Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the solution in the tank (kept thoroughly mixed by stirring) flows out at the constant rate of r_o liters per second.
To set up a differential equation for $x(t)$, we estimate the change Δx in x during the brief time interval $[t, t+\Delta t]$. The amount of solute that flows into the tank during Δt seconds is $r_i c_i \Delta t$ grams.
The amount of solute that flows out of the tank during the same time interval depends on the concentration $c_o(t)$ of solute in the solution at time t. But as noted in Fig. 2.1, $c_o(t) = \dfrac{x(t)}{v(t)}$, where $v(t)$ denotes the volume (not constant unless $r_i = r_o$) of solution in the tank at time t.

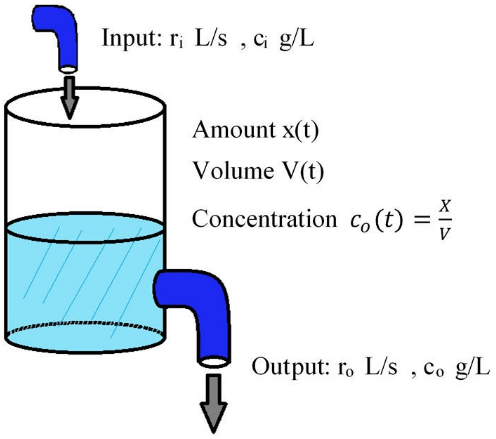

Figure 2.1: The single-tank mixture problem

Then

$$\Delta x = \text{grams input} - \text{grams output} \approx r_i c_i \Delta t - r_o c_o \Delta t.$$

We now divide by Δt:

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o.$$

Finally, we take the limit as $\Delta t \to 0$; if all the functions involved are continuous and $x(t)$ is differentiable, then the error in this approximation also approaches zero, and we obtain the differential equation

$$\frac{dx}{dt} = r_i c_i - r_o c_o, \tag{2.30}$$

in which r_i, c_i and r_o are <u>constants</u>, but c_o denotes the variable concentration

$$c_o(t) = \frac{x(t)}{v(t)} \tag{2.31}$$

of solute in the tank at time t. Thus the amount $x(t)$ of solute in the tank satisfies the differential equation

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{v} x, \tag{2.32}$$

which is a linear first-order differential equation for the amount $x(t)$ of solute in the tank at time t.

Example 4. Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (years), the pollutant concentration of Lake Erie (caused by past industrial pollution that has now been ordered to cease) is <u>five times</u> that of Lake Huron. If the outflow, henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to <u>twice</u> that of Lake Huron?

Solution: Here we have

$$v = 480 \; (km^3),$$

$$r_i = r_o = r = 350\ (\frac{km^3}{yr}),$$

$$c_i = c \text{ (the pollutant concentration of Lake Huron)}$$

$$\text{and} \quad \mathrm{x_0 = x(0) = 5\ cv},$$

and the question is this: When is $x(t) = 2\ cV$? with this notation, eq.(2.32) is the equation

$$\frac{dx}{dt} = rc - \frac{r}{v}\, x,$$

which we rewrite in the linear first-order form

$$\frac{dx}{dt} + \frac{r}{v}x = rc \tag{2.33}$$

with constant coefficients $p = \frac{r}{v}$, $q = rc$, and integrating factor $\mu = e^{(\frac{r}{v})\, t}$.
Then the solution of equation (2.33) is

$$x(t) = e^{-(\frac{r}{v})\, t}\Big(\int e^{(\frac{r}{v})\, t}\ rc\ dt\Big)$$

$$x(t) = e^{-(\frac{r}{v})\, t}\Big(e^{(\frac{r}{v})\, t}\ cv + \bar{c}\Big)$$

$$x(t) = cv + e^{-(\frac{r}{v})\, t}\ \bar{c}. \tag{2.34}$$

Substitute the initial condition $x(0) = 5\ cv$, gives

$$5\ cv = cv + \bar{c} \implies \bar{c} = 4\ cv.$$

Substitute the value of $\bar{c}$ in (2.34), yields

$$x(t) = cv + 4\ cv\ e^{-(\frac{r}{v})\, t}.$$

To find when $x(t) = 2\ cv$, we therefore need only solve the equation

$$cv + 4\ cv\ e^{-(\frac{r}{v})\, t} = 2\ cv \quad \text{for t.}$$

$$\implies t = \frac{v}{r}\ ln4 = \frac{480}{350}\ ln4 \approx 1.901\text{years}.$$

Example 5. A 120-gallon (gal) tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution: The interesting feature of this example is that due to the differing rates of inflow and outflow, the volume of brine in the tank increases steadily with $v(t) = 90 + t$ gallons. The change Δx in the amount x of salt in the tank from time t to time $t + \Delta t$ (minutes) is given by

$$\Delta x \approx (4)(2)\Delta t - 3(\frac{x}{90+t})\Delta t,$$

so our differential equation is

$$\frac{dx}{dt} + \frac{3}{90+t}\,x = 8.$$

An integrating factor is

$$\mu(x) = exp\Big(\int \frac{3}{90+t}\,dt\Big) = e^{3\,ln(90+t)} = (90+t)^3,$$

which gives

$$\frac{d}{dt}\Big((90+t)^3 x\Big) = 8(90+t)^3;$$
$$(90+t)^3 x = 2(90+t)^4 + c.$$

Substitution of $x(0) = 90$ gives $c = -(90)^4$, so the amount of salt in the tank at time t is

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

The tank is full after 30 min, and when $t = 30$, we have

$$x(30) = 2(90+30) - \frac{90^4}{120^3} \approx 202\ (lb)$$

of salt in the tank.